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# Integrability Conditions on Periodic Functions Related to their Fourier Transforms

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## 1. INTRODUCTION

Let $f$ be a periodic function and $c_n(f)$ its Fourier coefficients. We want to give integrability conditions on $f$ ensuring the convergence of the series

$$\sum{}' \, | c_n(f)| \, | n |^{\alpha-1}, \tag{1.1}$$

where $\alpha$ is a real number satisfying $0 \leqslant \alpha < 1$. (The symbol $\sum'$ means that the summation is carried out from $-\infty$ to $+\infty$ except for $n = 0$.) We assume the period to be 1. For $\alpha = 0$ there is a global condition, given by Zygmund [8, p. 242], saying that (1.1) converges if $|f| \log^+ |f| \in L$. There is a corresponding local condition by the same author [7, p. 299], saying that if $f$ is bounded outside every neighborhood of $x = 0$, even and nonincreasing in a right neighborhood of $x = 0$, then

$$\int_0^1 f(x) \log \frac{1}{x} \, dx < \infty \tag{1.2}$$

is necessary and sufficient for the convergence of (1.1). For $\alpha > 0$ there seem to exist only local conditions. Sz.-Nagy [4, p. 121] has proved that the convergence of

$$\int_0^1 f(x) \, x^{-\alpha} \, dx \tag{1.3}$$

is necessary and sufficient for the convergence of (1.1), if $f$ is odd or even, bounded outside a neighborhood of $x = 0$, and nonincreasing in $]0, \frac{1}{2}]$. For $0 < \alpha < \frac{1}{2}$ it is easily seen that we do not have to require $f$ to be monotone in the entire interval $]0, \frac{1}{2}]$ but only in $]0, \delta]$ for some $\delta > 0$. For $\alpha \geqslant \frac{1}{2}$ there is, however, no such generalization since there are continuous (and, thus, bounded) functions, such that (1.1) diverges ([3, Chapter IV: 6]). Boas [1, p. 14] has raised the question, whether the convergence of (1.2) or (1.3) is a sufficient condition to ensure the convergence of (1.1), when $f$ is positive in a right neighborhood of $x = 0$, bounded outside a neighborhood of $x = 0$ and odd or even. From Salem's result it is clear that this is not true for $\alpha \geqslant \frac{1}{2}$. For $0 \leqslant \alpha < \frac{1}{2}$ an attempt to answer the question has been made by Izumi and Izumi [2, p. 584]. Unfortunately, their counterexamples are functions that are not bounded at $x = \frac{1}{2}$ (or $x = \pi$ when $f$ has period $2\pi$).

In this paper we give in Section 3 a global condition for the convergence of (1.1) when $0 < \alpha < \frac{1}{2}$. The global conditions are used in Section 4 to deduce local criteria for $0 \leqslant \alpha < \frac{1}{2}$, and in Section 5 we give counterexamples showing that our conditions, in a way, are best possible. These examples also show that it is feasible to have (1.2) ($\alpha = 0$) or (1.3) ($0 < \alpha < \frac{1}{2}$) convergent, $f$ positive and (1.1) divergent. Section 2 contains a few lemmas that we have to use in more than one section.

## 2. Some Lemmas

Lemma 2.1. *Let $\sum_{p=1}^{\infty} a_p$ be a positive and convergent series and let $\delta$ be a positive number. Then there exists a sequence $(c_p)_1^{\infty}$ such that*

$$a_p \leqslant c_p, \qquad \sum_{p=1}^{\infty} c_p \text{ is convergent} \qquad \text{and} \qquad 2^{-\delta} \leqslant \frac{c_{p+1}}{c_p} \leqslant 2^{\delta}. \tag{2.1}$$

*Proof.* Since $a_{p+k} \to 0$ when $k \to \infty$, it is clear that for every $p \geqslant 1$ there exists an integer $k_p$, such that

$$b_p = \max_{k \geqslant 0} 2^{-\delta k} a_{p+k} = 2^{-\delta k_p} a_{p+k_p}.$$

Obviously, $b_p \geqslant a_p$ and $b_{p+k} = 2^{\delta k} b_p$, $k = 0, 1, \ldots, k_p$. The sequence $(b_p)_1^{\infty}$ satisfies the right inequality of (2.1) because

$$b_{p+1} = \max_{k \geqslant 0} 2^{-\delta k} a_{p+1+k} = 2^{\delta} \max_{k \geqslant 1} 2^{-\delta k} a_{p+k} \leqslant 2^{\delta} b_p.$$

If $(p_\nu)_1^{\infty}$ is the increasing sequence of integers such that $b_{p_\nu} = a_{p_\nu}$, then

$$\sum_{l=p_\nu+1}^{p_{\nu+1}} b_l \leqslant \sum_{k=0}^{p_{\nu+1}-p_\nu} 2^{-\delta k} b_{p_{\nu+1}} \leqslant \frac{2^{\delta}}{2^{\delta}-1} a_{p_{\nu+1}}.$$

Thus, the convergence of $\sum_{p=1}^{\infty} a_p$ implies the convergence of $\sum_{p=1}^{\infty} b_p$. Put

$$c_p = \max_{0 \leqslant k \leqslant p} 2^{-\delta k} b_{p-k} = 2^{-\delta k_p} b_{p-k_p}.$$

Evidently $c_p \geqslant b_p$ and $c_{p-k} = 2^{\delta k} c_p$, $k = 0, 1,..., k_p$. The sequence $(c_p)_1^\infty$ satisfies the left inequality of (2.1) because

$$2^{-\delta} c_p = \max_{0 \leqslant k \leqslant p} 2^{-\delta(k+1)} b_{p-k} = \max_{1 \leqslant k \leqslant p+1} 2^{-\delta k} b_{p+1-k} \leqslant c_{p+1} .$$

We find that either $c_{p+1} = 2^{-\delta} c_p$ or $c_{p+1} = b_{p+1}$. In the former case the right inequality of (2.1) is trivially true, and in the latter case we get

$$\frac{c_{p+1}}{c_p} = \frac{b_{p+1}}{c_p} \leqslant \frac{b_{p+1}}{b_p} \leqslant 2^\delta.$$

Thus, the sequence $(c_p)_1^\infty$ satisfies (2.1).

Consider now the increasing sequence of integers, $(p_\nu)_1^\infty$, such that $c_{p_\nu} = b_{p_\nu}$. Then

$$\sum_{l=p_\nu}^{p_{\nu+1}-1} c_l \leqslant \sum_{k=0}^{p_{\nu+1}-p_\nu} 2^{-\delta k} b_{p_\nu} \leqslant \frac{2^\delta}{2^\delta - 1} b_{p_\nu} .$$

Now $\sum_{p=1}^\infty c_p$ is convergent since $\sum_{p=1}^\infty b_p$ is convergent, and the proof is complete.

Using this lemma we prove a corresponding lemma for functions on $[0, \infty[$.

LEMMA 2.2. *Let $\delta$ be a positive number and let $g$ be a positive, integrable function on $[0, \infty[$ such that $a^x g(x)$ is decreasing or increasing for some $a > 0$. Then there exists an integrable function $g_2(x) \geqslant g(x)$ such that $2^{\delta x} g_2(x)$ is nondecreasing and $2^{-\delta x} g_2(x)$ is nonincreasing.*

*Proof.* We prove the lemma when $a^x g(x)$ is increasing. The decreasing case is treated analogously. We can, without loss of generality, assume that $a > 1$ and thus for $k \leqslant x \leqslant k + 1$ we have

$$\frac{g(k)}{a} \leqslant g(x) \leqslant a g(k+1). \tag{2.2}$$

Therefore, the integrability of $g(x)$ is equivalent to the convergence of the series $\sum_{k=1}^\infty g(k)$. By Lemma 2.1 we can construct a function $g_1$, defined on the positive integers, such that $g_1(k) \geqslant g(k)$, $\sum_{k=0}^\infty g_1(k)$ is convergent and

$$2^{-\delta} \leqslant \frac{g_1(k+1)}{g_1(k)} \leqslant 2^\delta. \tag{2.3}$$

Define $g_2$ in the interval $[k, k+1[$, $k = 0, 1, 2,...,$ by

$$g_2(k+x) = a 2^\delta g_1(k) \left(\frac{g_1(k+1)}{g_1(k)}\right)^x, \qquad 0 \leqslant x \leqslant 1.$$

Using (2.2) and (2.3) we obtain

$$g_2(k+x) \geqslant a2^\delta \min(g_1(k), g_1(k+1)) \geqslant ag_1(k+1) \\ \geqslant ag(k+1) \geqslant g(k+x).$$

Moreover, if $t \geqslant 0$, then

$$2^{-\delta t} \leqslant \frac{g_2(x+t)}{g_2(x)} \leqslant 2^{\delta t}.$$

This means that $2^{\delta x} g_2(x)$ is nondecreasing and $2^{-\delta x} g_2(x)$ is nonincreasing. Finally we observe that $g_2(k+x) \leqslant a2^\delta \max(g_1(k), g_1(k+1))$, and, therefore, $\sum_0^\infty g_2(k)$ converges. Consequently, $g_2$ is integrable and the proof is complete.

We also need the lemma formulated for functions on $]0, 1]$.

LEMMA 2.3. *Let $\delta$ be a positive number and let $g$ be a positive, integrable function on $]0, 1]$, such that $x^{-a}g(x)$ is decreasing for some real number $a$. Then there exists an integrable function $g_2(x) \geqslant g(x)$, such that $x^{1+\delta}g_2(x)$ is increasing and $x^{1-\delta}g_2(x)$ is decreasing.*

The lemma is easily proved by using the transformation $x \to 2^{-x}$ and Lemma 2.2.

LEMMA 2.4. *Let $h$ be a positive and continuous function on $[0, \infty[$ such that $h(t)\, 2^{\delta t}$ is nondecreasing and $h(t)\, 2^{-\delta t}$ is nonincreasing for some $\delta$ satisfying $0 < \delta < 1$. If $\alpha < 1$, then the equation*

$$h(\log^+ x 2^{(1-\alpha)k}) = 1/x \tag{2.4}$$

*has a unique solution $x = a_k$. If*

$$x_k = \log^+ a_k 2^{(1-\alpha)k},$$

*then there exists a natural number $k_0$ such that*

$$(1-\alpha)(1+\delta)^{-1} \leqslant x_{k+1} - x_k \leqslant (1-\alpha)(1-\delta)^{-1}, \qquad k = k_0, k_0+1, \ldots. \tag{2.5}$$

*Proof.* Putting $t = x2^{(1-\alpha)k}$ we see that the Eq. (2.4) is equivalent to the equation

$$h(\log^+ t) = 2^{(1-\alpha)k}/t. \tag{2.6}$$

We multiply each side in (2.6) by $t^\delta$. By hypothesis $t^\delta h(\log^+ t)$ is nondecreasing and $2^{(1-\alpha)k}t^{-1+\delta}$ is decreasing. Therefore, (2.6) has a unique solution

$N_k = a_k 2^{(1-\alpha)k}$. It is easily seen that the sequence $(N_k)_1^\infty$ is increasing. By hypothesis $h(\log^+ t)\, t^{-\delta}$ is nonincreasing, and using this fact combined with (2.6) we obtain

$$2^{(1-\alpha)(k+1)} N_{k+1}^{-(1+\delta)} \leqslant 2^{(1-\alpha)k} N_k^{-(1+\delta)}.$$

Hence,

$$\frac{N_{k+1}}{N_k} \geqslant 2^{(1-\alpha)/(1+\delta)}. \tag{2.7}$$

In particular, it follows that $N_k \geqslant 1$, and, thus,

$$x_{k+1} - x_k = \log \frac{N_{k+1}}{N_k} \qquad \text{for } k \geqslant k_0\,. \tag{2.8}$$

From (2.6) and the fact that $t^\delta h(\log^+ t)$ is nondecreasing we deduce

$$\frac{N_{k+1}}{N_k} \leqslant 2^{(1-\alpha)/(1-\delta)}. \tag{2.9}$$

Combining (2.7), (2.8), and (2.9) we get (2.5). The proof is complete.

## 3. Global Conditions

In this section we shall state a theorem (in two equivalent forms) which imposes integrability conditions on a periodic function $f$ ensuring the convergence of the series $\sum' |\, c_n(g)|\, |\, n\,|^{\alpha-1}$ for every periodic function $g$ such that $|\, g\,| \leqslant |f|$.

Let $F$ be an indefinite integral of a periodic and integrable function $f$. We denote by $\omega(F, h)$ the modulus of continuity of $F$, and the quadratic modulus of continuity, $\omega_2(F, h)$, is defined by

$$\omega_2(F, h) = \sup_{0 \leqslant t \leqslant h} \left( \int_0^1 |\, F(x+t) - F(x)|^2\, dx \right)^{1/2}.$$

We normalize $f$, such that $\int_0^1 |\, f\,|\, dx = 1$, and estimate $\omega_2(H_a\,, h)$, where

$$H_a(x) = \int_0^x f(t)\, dt, \qquad |\, f\,| > a.$$

We choose $t$ satisfying $0 \leqslant t \leqslant h$ and obtain

$$\int_0^1 |\, H_a(x+t) - H_a(x)|^2\, dx \leqslant \omega(H_a\,, h) \int_0^1 |\, H_a(x+t) - H_a(x)|\, dx.$$

It is easily seen that $\omega(H_a, h) \leqslant V_a$ and

$$\int_0^1 |H_a(x+t) - H_a(x)|\, dx \leqslant V_a t,$$

where $V_a$ is the total variation on [0, 1] of $H_a$. Thus,

$$\omega_2(H_a, h) \leqslant V_a h^{1/2}. \tag{3.1}$$

We also need another property of $\omega_2$, namely

$$\omega_2(F + G, h) \leqslant \omega_2(F, h) + \omega_2(G, h). \tag{3.2}$$

This inequality follows from Minkowski's inequality and the definition of $\omega_2$ as a supremum.

For the proof of Theorem 3.3 we need two lemmas.

LEMMA 3.1. *Let $f$ be a periodic function, with period* 1, *and let $F$ be an indefinite integral of $f$. Suppose that*

$$\sum_{p=1}^{\infty} 2^{p(\alpha+1/2)} \omega_2(F, 2^{-p})$$

*is convergent for some $\alpha$ satisfying $0 \leqslant \alpha < \frac{1}{2}$. Then the series $\sum' |c_n(f)|\, |n|^{\alpha-1}$ is convergent.*

*Proof.* We choose a constant $K$ such that $L(x) = F(x) + Kx$ is periodic. If $n \neq 0$ it is known that $L(x)$ has Fourier coefficients

$$c_n(L) = c_n(f) \cdot (2\pi i n)^{-1}. \tag{3.3}$$

It is also known (e.g., [8, p. 241]) that

$$\sum_{n=2^{p-1}+1}^{2^p} |c_n(L)| \leqslant 2^{p/2} \omega_2(L, 2^{-p-1}), \qquad p = 1, 2, \ldots.$$

Therefore,

$$\sum_{n=2^{p-1}+1}^{2^p} |c_n(L)|\, |n|^{\alpha} \leqslant 2^{p(\alpha+1/2)} \omega_2(L, 2^{-p-1}).$$

Thus using (3.2) we obtain

$$\sum_{n=2^{p-1}+1}^{2^p} |c_n(L)|\, |n|^{\alpha} \leqslant 2^{p(\alpha+1/2)} \omega_2(F, 2^{-p-1}) + K 2^{p(\alpha-1/2)}, \qquad p = 1, 2, \ldots . \tag{3.4}$$

The same estimations are valid for negative values of $n$, and, therefore, the lemma follows from (3.3) and (3.4).

*Remark.* Applying the lemma to a bounded function we see that if $\alpha$ is a real number satisfying $0 \leqslant \alpha < \frac{1}{2}$, and if $f$ is a bounded and periodic function, then the series $\sum' | c_n(f)| \, | n |^{\alpha-1}$ is convergent.

LEMMA 3.2. *Let $f$ be a function with period* 1. *If*

$$E = \{x \mid 0 \leqslant x \leqslant 1 \text{ and } |f| \leqslant A\},$$

*and if*

$$H(x) = \int_{[0,x]\cap E} f(u)\, du,$$

*then*

$$\omega_2(H, h) \leqslant Ah(mE)^{1/2}.$$

*Proof.* By the definitions of $\omega_2$ and $\omega$, it follows that

$$\omega_2{}^2(H, h) \leqslant \omega(H, h)\, V_H h, \tag{3.5}$$

where $V_H$ is the total variation of $H$. However,

$$\omega(H, h) \leqslant hA \tag{3.6}$$

and

$$V_H \leqslant AmE. \tag{3.7}$$

Combining (3.5)–(3.7) we obtain the desired result.

THEOREM 3.3. *Let $f$ be a function with period* 1. *If $\alpha$ is a real number satisfying* $0 < \alpha < \frac{1}{2}$, *and if there exists a positive and continuous function $h_1$, on* $[0, \infty[$ *such that*

$$h_1(x)\, x^k \text{ is increasing for some constant } k, \tag{3.8}$$

$$\int_0^\infty \frac{dx}{h_1(x)} < \infty \tag{3.9}$$

*and*

$$\int_0^1 |f|\, (h_1(|f|))^{\alpha/(1-\alpha)}\, dx < \infty, \tag{3.10}$$

*then*

$$\sum{}' \, | c_n(g)| \, | n |^{\alpha-1} < \infty$$

*for every function $g$ with period* 1 *such that* $|g| \leqslant |f|$.

THEOREM 3.3′. *Let $f$ be a function with period 1. If $\alpha$ is a real number satisfying $0 < \alpha < \frac{1}{2}$, and if there exists a positive and continuous function $h$ on $[0, \infty]$ such that*

$$h(x)\, c^x \text{ is increasing for some positive constant } c, \tag{3.11}$$

$$\int_0^\infty \frac{dx}{h(x)} < \infty \tag{3.12}$$

*and*

$$\int_0^1 |f|^{1/(1-\alpha)} (h(\log^+ |f|))^{\alpha/(1-\alpha)}\, dx < \infty, \tag{3.13}$$

*then*

$$\sum{}' \,|c_n(g)|\, |n|^{\alpha-1} < \infty$$

*for every function $g$ with period 1 such that $|g| \leqslant |f|$.*

Theorem 3.3 is easier than Theorem 3.3′ to apply to an individual function, but Theorem 3.3′ better shows the structure of the theorem and has a proof which is more easy to survey.

*Proof.* If $h$ satisfies (3.11), (3.12) and (3.13) when $\log = {}^a\!\log$ for some $a > 1$, then $h_3(x) = h(x\, {}^a\!\log 2)$ satisfies (3.11), (3.12) and (3.13) when $\log = {}^2\!\log$. Thus, we may, without loss of generality, assume that $\log = {}^2\!\log$. Now putting $h(x) = 2^{-x} h_1(2^x)$, some computation establishes the equivalence of Theorems 3.3 and 3.3′. Because of hypothesis (3.11) we can apply Lemma 2.2 to the function $g(x) = 1/h(x)$. It is easy to see that the modified function $h_2(x) = 1/g_2(x)$ satisfies (3.11), (3.12), and (3.13). Therefore, choosing $\delta$ such that $0 < \delta < 1$ we can, still without loss of generality, suppose that $h(x)\, 2^{\delta x}$ is nondecreasing and $h(x)\, 2^{-\delta x}$ is nonincreasing. The choice of $\delta$ makes the function $t(h(\log^+ t))^\alpha$ increasing in $[0, \infty[$, and, thus, if (3.13) is valid for a function $f$, then (3.13) is also valid for all functions $g$ such that $|g| \leqslant |f|$. Therefore, it is enough to prove the theorem with $g = f$.

Put

$$N_k = a_k 2^{(1-\alpha)k} \qquad \text{and} \qquad x_k = \log^+ N_k\,, \qquad k = 1, 2, \ldots,$$

where $a_k$ are the unique solutions of the equation

$$h(\log^+ x 2^{(1-\alpha)k}) = 1/x.$$

From (2.7) and (2.9) it follows that

$$2^{(1-\alpha)/(1+\delta)} \leqslant \frac{N_{k+1}}{N_k} \leqslant 2^{(1-\alpha)/(1-\delta)}, \qquad k = 1, 2, \ldots. \tag{3.14}$$

Let $F$ be an indefinite integral of $f$, and for $k = 1, 2, \ldots$, and $N_0 = 0$ we put

$$E_a = \{x \mid 0 \leqslant x \leqslant 1, |f(x)| > a\},$$

and

$$V_a = \int_{E_a} |f| \, dx, \qquad F(x) = G_a(x) + H_a(x),$$

where

$$H_a(x) = \int_{E_a \cap [0,x]} f(t) \, dt,$$

$$S_k = E_{N_{k-1}} \backslash E_{N_k} \qquad \text{and} \qquad F_k(x) = H_{N_{k-1}}(x) - H_{N_k}(x).$$

The sequence $(N_k)_1^\infty$ is increasing, and, therefore, the convergence of

$$\int_0^1 |f|^{1/(1-\alpha)} (h(\log^+ |f|))^{\alpha/(1-\alpha)} \, dx$$

is equivalent to the convergence of the series

$$\sum_{k=1}^{\infty} \int_{S_k} |f|^{1/(1-\alpha)} (h(\log^+ |f|))^{\alpha/(1-\alpha)} \, dx.$$

By (3.14) and the fact that $t \, (h(\log^+ t))^\alpha$ is increasing for $t \geqslant 0$, we obtain that this series is convergent if and only if

$$\sum_{k=1}^{\infty} 2^k a_k^{1/(1-\alpha)} (h(x_k))^{\alpha/(1-\alpha)} \, m(S_k) = \sum_{k=1}^{\infty} 2^k a_k m(S_k) < \infty. \tag{3.15}$$

Thus, hypothesis (3.13) is equivalent to (3.15). Because of Lemma 3.1, it is sufficient to prove that

$$\sum_{p=1}^{\infty} 2^{p(\alpha+1/2)} \omega_2(F, 2^{-p}) < \infty, \tag{3.16}$$

where $F$ is an indefinite integral of $f$. Applying Lemma 3.2 to $F_k$ we get

$$\omega_2(F_k, 2^{-p}) \leqslant N_k (m(S_k))^{1/2} \, 2^{-p}. \tag{3.17}$$

We observe that $G_{N_p} = \sum_{k=1}^{p} F_k$, and, therefore, using (3.2) and (3.17), we obtain

$$\omega_2(G_{N_p}, 2^{-p}) \leqslant \sum_{k=1}^{p} \omega_2(F_k, 2^{-p}) \leqslant \sum_{k=1}^{p} N_k (m(S_k))^{1/2} \, 2^{-p}.$$

Thus,

$$\sum_{p=1}^{\infty} 2^{p(\alpha+1/2)}\omega_2(G_{N_p}, 2^{-p}) \leqslant \sum_{p=1}^{\infty} 2^{p(\alpha+1/2)} \sum_{k=1}^{p} N_k(m(S_k))^{1/2} 2^{-p}$$

$$= \sum_{k=1}^{\infty} N_k(m(S_k))^{1/2} \sum_{p=k}^{\infty} 2^{p(\alpha-1/2)} \leqslant K_0 \sum_{k=1}^{\infty} N_k(m(S_k))^{1/2} 2^{k(\alpha-1/2)} \tag{3.18}$$

$$= K_0 \sum_{k=1}^{\infty} 2^{k/2} a_k (m(S_k))^{1/2},$$

where $K_0$ is a positive constant. By Cauchy's inequality

$$\sum_{k=1}^{\infty} 2^{k/2} a_k (m(S_k))^{1/2} \leqslant \left(\sum_{k=1}^{\infty} 2^k a_k m(S_k)\right)^{1/2} \left(\sum_{k=1}^{\infty} a_k\right)^{1/2}. \tag{3.19}$$

If $n$ is a positive integer, and $t$ is a real number satisfying $0 \leqslant t \leqslant 1$, then it follows from the uniform growth condition of $h$ that

$$\frac{2^{-\delta}}{h(n-t)} \leqslant \frac{1}{h(n)} \leqslant \frac{2^{\delta}}{h(n+t)}. \tag{3.20}$$

A consequence of (3.20) is that $\int_0^\infty dx/h(x)$ converges if and only if $\sum_{n=0}^{\infty} 1/h(n)$ converges. By Lemma 2.4 we have $x_{k+1} - x_k \geqslant c > 0$ for $k \geqslant k_0$, and, therefore, there is at most $[1/c] + 1$ of the numbers $x_k$ in an interval $[n, n+1]$. Using (3.20) we conclude that if $\sum_{k=0}^{\infty} 1/h(k)$ is convergent, then $\sum_{k=1}^{\infty} 1/h(x_k)$ is convergent. Now combining (3.13), (3.15), (3.18), and (3.19), we see that

$$\sum_{p=1}^{\infty} 2^{p(\alpha+1/2)}\omega_2(G_{N_p}, 2^{-p}) < \infty. \tag{3.21}$$

From the estimation (3.1) we observe that

$$\sum_{p=1}^{\infty} 2^{p(\alpha+1/2)}\omega_2(H_{N_p}, 2^{-p}) \leqslant \sum_{p=1}^{\infty} 2^{p\alpha} V_{N_p}. \tag{3.22}$$

We use Abelian transformation on the series $\sum_{p=1}^{\infty} 2^{p\alpha} V_{N_p}$, and deduce that it converges exactly when $\sum_{p=1}^{\infty} 2^{p\alpha}(V_{N_p} - V_{N_{p+1}})$ converges. An easy computation shows that the latter series converges and diverges exactly as $\sum_{k=1}^{\infty} 2^k a_k m(S_k)$. Therefore, by (3.13), (3.15), and (3.22), we infer that

$$\sum_{p=1}^{\infty} 2^{p(\alpha+1/2)}\omega_2(H_{N_p}, 2^{-p}) < \infty. \tag{3.23}$$

Finally (3.16) follows from (3.2), (3.21), (3.23) and the fact that

$$F(x) = G_{N_p} + H_{N_p}$$

for every $p$. The proof is complete.

*Remark* 1. The hypothesis (3.9) is not superfluous. We show this later as Theorem 5.4.

COROLLARY. *Let $\alpha$ be a real number satisfying $0 < \alpha < \frac{1}{2}$, and let $f$ be a periodic function, with period* 1. *If*

$$\int_0^1 |f|^{1/(1-\alpha)} (\log^+ |f|)^{\alpha/(1-\alpha)+\epsilon} \, dx$$

*is convergent for any $\epsilon > 0$, then the series $\sum' | c_n(g)| \, | n |^{\alpha-1}$ converges, where $g$ is a periodic function such that $| g | \leqslant | f |$.*

*Proof.* Choose $h(t) = t^{1+(\epsilon(1-\alpha)/\alpha)}$ and apply Theorem 3.3′.

*Remark* 2. By Remark 1 we see that the corollary is false with $\epsilon = 0$.

*Remark* 3. Choosing

$$h(t) = t(\log t)^{1+(\epsilon(1-\alpha)/\alpha)},$$

$$h(t) = t \log t(\log \log t)^{1+(\epsilon(1-\alpha)/\alpha)},\ldots,$$

etc. for $\epsilon > 0$ we obtain corollaries analogous to the corollary above.

*Remark* 4. The corresponding global condition for $\alpha = 0$ has been given by Zygmund [8, p. 242], and reads: "If $| f | \log^+ | f | \in L$, then $\sum' | c_n(f)| \, | n |^{-1} < \infty$." This condition is also necessary in a class of functions [6, p. 19–22].

*Remark* 5. For $\alpha \geqslant \frac{1}{2}$ Salem has shown [3, Chapter IV: 6] that there exists a continuous function $f$, such that $\sum' | c_n(f)| \, | n |^{\alpha-1} = \infty$. See also Zygmund [8, p. 225–228]. Therefore, Theorem 3.3 is not adequate for $\alpha \geqslant \frac{1}{2}$.

## 4. LOCAL CONDITIONS

In this section we study functions, with period 1, that are bounded on $[\delta, 1 - \delta]$ for every $\delta > 0$. The functions only misbehave in a neighborhood of $x = 0$. The following lemma enables us to deduce a local condition from the global condition in the preceeding section.

LEMMA 4.1. *Let $\alpha$ be a real number, $0 < \alpha < 1$, and let $f$ be a function on $]0, 1[$, such that for some real $a$ $x^{-a}\,|f(x)|$ is decreasing in a right neighborhood of $x = 0$ and bounded elsewhere, and such that $x^{-\alpha}f(x) \in L[0, 1]$. Then there exists a positive and continuous function $h$, defined on $[0, \infty[$ and a positive number $K$, $K < 2^{1/\alpha}$, such that*

(a) $K^{-x}h(x)$ *is nonincreasing,*

(b) $K^{x}h(x)$ *is nondecreasing,*

(c) $\int_0^\infty dx/h(x) < \infty$ *and*

(d) $\int_0^1 |f|^{1/(1-\alpha)}\,(h(\log^+|f|))^{\alpha/(1-\alpha)}\,dx < \infty.$

*Proof.* The growth and integrability conditions on $f$ imply by Lemma 2.3 the existence, for every $\delta > 0$, of a function $g(x) \geqslant 1$, such that $|f(x)| \leqslant g(x)$,

$$x^{1-\alpha+\delta}g(x) \text{ is nondecreasing,} \tag{4.1}$$

$$x^{1-\alpha-\delta}g(x) \text{ is nonincreasing,} \tag{4.2}$$

and

$$\int_0^1 \frac{g(x)}{x^\alpha}\,dx < \infty.$$

This integral, however, converges if and only if

$$\sum_{p=0}^{\infty} g(2^{-p})\,2^{p(\alpha-1)} < \infty. \tag{4.3}$$

Put $a_p = g(2^{-p})\cdot 2^{p(\alpha-1)}$, and define the function $h$ at the points $x_p = \log g(2^{-p})$ by $h(x_p) = 1/a_p$, $p = 0, 1, 2,\ldots$. It now follows from (4.1) and (4.2) that

$$g(2^{-p})\,2^{(1-\alpha-\delta)x} \leqslant g(2^{-p-x}) \leqslant g(2^{-p})\,2^{(1-\alpha+\delta)x}, \qquad 0 \leqslant x \leqslant 1. \tag{4.4}$$

This gives

$$(1-\alpha-\delta) \leqslant x_{p+1} - x_p \leqslant (1-\alpha+\delta), \tag{4.5}$$

and from (4.4) and the definition of $h(x_p)$,

$$2^{-\delta} \leqslant \frac{h(x_p)}{h(x_{p+1})} = \frac{g(2^{-p-1})}{g(2^{-p})\cdot 2^{1-\alpha}} \leqslant 2^{\delta}.$$

Therefore, $h$ can be extended to a continuous function, defined for $x \geqslant 0$, and satisfy (a) and (b) with $K = 2^{\delta c}$, where $c = (1-\alpha-\delta)^{-1} + 1$. Thus, choosing $\delta$ small enough we obtain $K < 2^{1/\alpha}$. By (4.3)

$$\sum_{p=0}^{\infty} \frac{1}{h(x_p)} = \sum_{p=0}^{\infty} a_p$$

is convergent, and from the right inequality of (4.5), we conclude that

$$\sum_{k=1}^{\infty} \frac{1}{h(k)} \qquad \text{and, thus,} \qquad \int_0^1 \frac{dx}{h(x)}$$

are convergent. Condition (c) of the lemma is, therefore, satisfied. From the growth properties of $g$ and $h$ we deduce

$$\begin{aligned}\int_0^1 & (g(x))^{1/(1-\alpha)} (h(\log^+ g))^{\alpha/(1-\alpha)}\, dx \\ &\leqslant K^{1/(1-\alpha)} \sum_{p=0}^{\infty} g(2^{-p})^{1/(1-\alpha)}\, 2^{\delta\alpha/(1-\alpha)} (h(x_p))^{\alpha/(1-\alpha)}\, 2^{-p} \\ &= K^{1/(1-\alpha)} 2^{\delta\alpha/(1-\alpha)} \sum_{p=0}^{\infty} a_p\,.\end{aligned}$$

It now follows from (4.3) that condition (d) of the lemma is satisfied by the function $g$. However, $K < 2^{1/\alpha}$ makes $t(h(\log^+ t))^\alpha$ a nondecreasing function of $t$. We conclude that condition (d) is satisfied also for the function $f$, and the lemma is proved.

We define $f^*$ to be the smallest function with period 1, satisfying $|f| \leqslant f^*$ and having the property that $f^*$ is nonincreasing in $]0, \frac{1}{2}]$ and nondecreasing in $[\frac{1}{2}, 1[$. We then have the following.

THEOREM 4.2. *The following implication holds for* $0 < \alpha < \frac{1}{2}$

$$\int_{-1/2}^{1/2} \frac{f^*}{|x|^\alpha}\, dx < \infty \Rightarrow \sum{}' \,| c_n(f)|\, |n|^{\alpha-1} < \infty. \tag{4.6}$$

*Proof.* In the above proof we have in fact shown the lemma to be true for a nonincreasing function $g$, that majorizes $|f|$. It is then true for any smaller function and finite sums of such functions, in particular for the function $f^*$. For $0 < \alpha < \frac{1}{2}$ the conclusion of the theorem now is an immediate consequence of Theorem 3.3.

COROLLARY 4.3. *The conditions on $f$ posed in Theorem* 3.3 *are necessary and sufficient when* $0 < \alpha < \frac{1}{2}$ *for the convergence of* $\sum_{n=1}^{\infty} |c_n(f)|\, |n|^{\alpha-1}$ *if $f$ is real, nonincreasing in a right neighborhood of* $x = 0$, *bounded in* $[\delta, 1 - \delta]$ *for every* $\delta > 0$, *and odd or even.*

*Proof.* This follows immediately from Lemma 4.1, Theorem 3.3, and the fact [4, p. 121] that the implication (4.6) is an equivalence for integrable, odd or even functions, nonincreasing in a right neighborhood of $x = 0$.

*Remark* 1. Implication (4.6) does not hold in general when $\alpha \geqslant \frac{1}{2}$. Salem has shown ([3, Chapter IV: 6] or [8, p. 225]), that there are continuous functions on [0, 1] such that

$$| c_n(f)| = \frac{1}{n^{1/2} \log n}.$$

His functions are, for example, of the form

$$f(x) = \sum_2^\infty \frac{e^{2\pi i(g(n)+nx)}}{n^{1/2} \log n},$$

for a certain real function $g(n)$. The functions $\frac{1}{2}\,\mathrm{Re}(f(x) + f(-x))$ or $\frac{1}{2}\,\mathrm{Im}(f(x) - f(-x))$ are bounded, real, even or odd and their cosine or sine coefficients are $\cos(2\pi g(n))\,(n^{1/2} \log n)^{-1}$. It is easy to see that this is also true for the corresponding functions, where $g(n)$ has been replaced by $tg(n)$, $t$ being real and nonzero:

$$f_t(x) = \sum_1^\infty \frac{\cos(2\pi t g(n))}{n^{1/2} \log n} \cos 2\pi n x,$$

$$g_t(x) = \sum_1^\infty \frac{\cos(2\pi t g(n))}{n^{1/2} \log n} \sin 2\pi n x.$$

Applying Beppo–Levi's theorem we find that there exists a number $t > 0$ such that

$$\sum_1^\infty \frac{| \cos(2\pi t g(n))|}{n \log n} = \infty,$$

but $f_t(x)$ and $g_t(x)$ are bounded, real, even and odd functions, respectively.

*Remark* 2. The condition on $f^*$ given in the theorem cannot, in general, be weakened. This is shown by a counterexample (Theorem 5.2).

The corresponding theorem in the case that $\alpha = 0$ is trivial. We list it here mainly because our counterexample (Theorem 5.3) shows that the theorem is almost as sharp as possible.

THEOREM 4.4. *If $f$ has period* 1, *is bounded in* $[\delta, 1 - \delta]$ *for every* $\delta > 0$, *and* $\lim_{x\to 0} x^a f(x) = 0$ *for some real number* $a$, *then the following implication holds:*

$$\int_{-1/2}^{1/2} | f(x)| \log \frac{1}{x}\, dx < \infty \Rightarrow \sum{}' \, | c_n(f)| \, | n |^{-1} < \infty. \tag{4.7}$$

*Proof.* By assumption $|f(x)| < A|x|^{-a}$ in some neighborhood of $x = 0$ and for some constant $A$. This gives

$$\log^+ |f| \leqslant \log^+ A + a \log 1/x.$$

It follows that $|f| \log^+ |f| \in L$, and the conclusion of the theorem, then, is a consequence of a well known theorem by Zygmund [8, p. 242].

## 5. Counterexamples

This section is used to show that Theorems 3.3 and 4.2 are, in a way, best possible, and that Theorem 4.4 is close to being best possible. In order to construct a counterexample corresponding to Theorem 4.2, we need a lemma.

Lemma 5.1. *If $0 \leqslant a_p \leqslant 1$, $p = 1, 2,\ldots$, and $\sum_1^\infty a_p$ diverges, then, for every $\delta > 0$, there exists an increasing sequence $(p_v)_1^\infty$ of positive integers, such that*

$$(1 + \delta)^{p_v - p_{v+1}} \leqslant \frac{a_{p_v}}{a_{p_{v+1}}} \leqslant (1 + \delta)^{p_{v+1} - p_v}$$

*and such that the series $\sum_{v=1}^\infty a_{p_v}$ diverges.*

*Proof.* A proof is found in Wik [5, p. 75].

Theorem 5.2. *Suppose that $0 < \alpha < 1$, and let $g$ be a nonincreasing positive function on $]0, 1]$, such that*

$$\int_0^1 \frac{g(x)}{x^\alpha}\, dx = \infty. \tag{5.1}$$

*Then there exists a function $f$, with period 1, such that $0 \leqslant f(x) \leqslant g(x)$ and such that*

(a) $\int_0^1 f(x)/x^\alpha\, dx < \infty$ *and*

(b) $\sum' |c_n(f)|\, |n|^{\alpha - 1} = \infty$.

*Proof.* Choose a positive number $c$, so large that

$$c^\alpha > 10 \qquad \text{and} \qquad c^{1-\alpha} > 10. \tag{5.2}$$

Put $a_p = g(c^{-p+1})\, c^{p(\alpha-1)}$, $p = 1, 2,\ldots$. It follows from (5.1) that the series $\sum_{p=1}^\infty a_p$ is divergent, and we may, without loss of generality, assume that $a_p < 1$. According to Lemma 5.1, there is a sequence $(p_v)_1^\infty$ such that

$$\left(\frac{4}{5}\right)^{p_{v+1} - p_v} < \frac{a_{p_v}}{a_{p_{v+1}}} < \left(\frac{5}{4}\right)^{p_{v+1} - p_v} \tag{5.3}$$

and such that $\sum_{v=1}^{\infty} a_{p_v}$ is divergent. We now choose a sequence $(b_v)_1^\infty$, $b_v < 1$, $v = 1, 2, \ldots$, such that

$$\sum_{v=1}^{\infty} b_v a_{p_v} < \infty \tag{5.4}$$

and

$$\sum_{v=1}^{\infty} b_v^{1-\alpha} a_{p_v} = \infty. \tag{5.5}$$

It is possible to choose $(b_v)_1^\infty$ as a nonincreasing sequence. Our function $f$ now is defined in the following way:

$$f(x) = \begin{cases} a_{p_v} c^{p_v(1-\alpha)} & \text{on} \quad I_k = [c^{-p_v}, c^{-p_v}(1 + b_v)], \qquad v = 2, 3, \ldots, \\ 0 & \text{elsewhere on } [0, 1]. \end{cases}$$

We then have $0 \leqslant f(x) \leqslant g(x)$ on $]0, 1[$ and

$$\int_0^1 \frac{f(x)}{x^\alpha}\, dx \leqslant \sum_{v=1}^{\infty} a_{p_v} \cdot b_v,$$

which converges by (5.4). Condition (a) of the theorem is, thus, satisfied. For the Fourier coefficients we use the following estimation:

$$\left| \int_{I_k} e^{2\pi i n x}\, dx \right| \begin{cases} \geqslant (2/\pi)\, mI_k, & |n|\, mI_k \leqslant \frac{1}{2} \\ \leqslant mI_k, & \text{every } n \\ \leqslant 1/\pi n, & |n|\, mI_k \geqslant \frac{1}{2}. \end{cases}$$

In the interval $c^{p_v}/3b_v \leqslant n \leqslant c^{p_v}/2b_v$ we, therefore, obtain

$$|c_n(f)| \geqslant \frac{2}{\pi} a_{p_v} c^{-p_v \alpha} b_v - \sum_{k=v+1}^{\infty} a_{p_k} c^{-p_k \alpha} b_k - \frac{1}{\pi n} \sum_{k=1}^{v-1} a_{p_k} c^{p_k(1-\alpha)}.$$

By (5.2) and (5.3), the terms of the first series decrease at least as those of a geometric series with ratio $\frac{1}{8}$. In the same way we find that the terms of the second series increase at least as those of a geometric series with ratio 8. Thus, for

$$n_v = c^{p_v}/3b_v \leqslant n \leqslant c^{p_v}/2b_v = n_v'$$

$$|c_n(f)| \geqslant a_{p_v} c^{-p_v \alpha} b_v \left( \frac{2}{\pi} - \frac{2}{7} \right).$$

Summation over these values of $n$ gives

$$\sum_{n_v}^{n_v'} |c_n(f)|\, n^{\alpha-1} \geqslant K a_{p_v} b_v^{1-\alpha}.$$

Since $n_{v+1} > n_v'$, it follows from (5.5) that the series $\sum' |c_n(f)|\,|n|^{\alpha-1}$ diverges. Thus, condition (b) is satisfied, and the theorem is proved.

*Remark.* By considering real and imaginary parts separately, it is easily seen that the preceding function, extended to an even or odd function, gives an example of a function which is bounded outside a neighborhood of $x = 0$, nonnegative in a right neighborhood of $x = 0$, and such that

$$\int_0^1 \frac{|f(x)|}{x^\alpha}\,dx < \infty \qquad \text{but} \qquad \sum_{n=1}^{\infty} |a_n(f)|\, n^{\alpha-1} = \infty$$

or

$$\sum_{n=1}^{\infty} |b_n(f)|\, n^{\alpha-1} = \infty,$$

respectively. This is an answer to a question raised by Boas [1, p. 14].

The corresponding counterexample for $\alpha = 0$ is the following.

THEOREM 5.3. *Let $g$ be a bounded function on $[\delta, 1]$, for every $\delta > 0$, satisfying*

$$\varliminf_{x\to 0+} a^{-1/x} g(x) > 0,$$

*for some real number $a > 1$. Then there exists a function $f$ with period 1, such that $0 \leqslant f(x) \leqslant g(x)$,*

$$\int_0^1 f(x) \log \frac{1}{x}\,dx < \infty \qquad \textit{and} \qquad \sum{}' |c_n(f)|\,|n|^{-1} = \infty. \tag{5.6}$$

*Proof.* Without loss of generality we may assume that $g(x) > a^{1/x}$ in a neighborhood to the right of $x = 0$.

Put $f(x) = 0$ except that $f(x) = 3^{-p}2^{3^p}$ on intervals of length $2^{-3^p}$ and right endpoints at $(3^p - p\log 3)^{-1} \log a$, $p = p_0, p_0 + 1, \ldots$. If $p_0$ is large enough, we have

$$0 \leqslant f(x) \leqslant g(x), \qquad \text{and} \qquad \int_0^1 f(x) \log \frac{1}{x}\,dx$$

converges as the series $\sum_{p=1}^{\infty} p\, 3^{-p}$. For

$$n_p = 5 \cdot 2^{3^{p-1}} \leqslant \tfrac{1}{2}\, 2^{3^p} = n_p'$$

we obtain, in analogy with the proof of Theorem 5.2, the following estimation for the Fourier coefficients of $f$:

$$|c_n(f)| \geqslant \frac{2}{\pi}\, 3^{-p} - \sum_{p+1}^{\infty} \frac{1}{3^k} - \frac{1}{\pi n} \sum_{p_0}^{p-1} 3^{-k} 2^{3^k}.$$

This gives

$$\sum_{n_p}^{n_p'} | c_n(f)| \, | n |^{-1} \geqslant 2/100,$$

and since $n_{p+1} \geqslant n_p'$ we conclude that $\sum' | c_n(f)| \, | n |^{-1}$ is divergent. The theorem is thereby proved.

*Remark.* Also this time we can extend the function to be even or odd and keep the properties (5.6).

By Zygmund [7, p. 299] (see also Boas [1, p. 13]), the integral and the series of (5.6) are equiconvergent for the class of even, integrable functions that are bounded in $[\delta, 1 - \delta]$ for every $\delta > 0$, and nonincreasing in a right neighborhood of the origin. The above example shows that this is no longer true if we replace the assumption that $f$ is monotone by the assumption that $f \geqslant 0$ in a neighborhood of $x = 0$. However, in the class of positive, integrable functions $f$ such that $f(x) = O(1/| x |)$ in a neighborhood of $x = 0$ and bounded in $[\delta, 1 - \delta]$ for every $\delta > 0$, the following equivalence holds:

$$| f | \log^+ | f | \in L[0, 1] \Leftrightarrow \sum' | c_n(f)| \, | n |^{-1} < \infty.$$

See Wik [6, p. 19].

Finally we give a counterexample showing that the condition of Theorem 3.3 that $1/h(x)$ is integrable cannot be weakened.

THEOREM 5.4. *Let $h$ be a positive function on $[0, \infty]$, such that $2^{\delta x}h(x)$ is increasing and $2^{-\delta x}h(x)$ is decreasing for some number $\delta$, $0 < \delta < 1$. Suppose further that*

$$\int_0^\infty \frac{dx}{h(x)} = \infty.$$

*Then there exists, for every $\alpha$ satisfying $0 < \alpha < 1$, a nonnegative function $f$ with period 1 such that*

$$\int_0^1 | f(x)|^{1/(1-\alpha)} \, (h(\log^+ | f |))^{\alpha/(1-\alpha)} \, dx < \infty,$$

*but*

$$\sum' | c_n(f)| \, | n |^{\alpha-1} = \infty.$$

*Proof.* Let $a_k$ be the unique solution of

$$h({}^{\mathrm{c}}\log^+ x c^{k(1-\alpha)}) = x^{-1}, \qquad k = 1, 2, \ldots.$$

By Lemma 2.4, applied with the base of the logarithm being $c$ instead of 2 it follows that if

$$x_k = {}^c\log^+ a_k c^{k(1-\alpha)},$$

then

$$(1-\alpha)(1+\delta)^{-1} < x_{k+1} - x_k < (1-\alpha)(1-\delta)^{-1}, \qquad k = k_0, k_0+1, \ldots \tag{5.7}$$

As in Theorem 5.2, we assume $c^\alpha$ and $c^{1-\alpha}$ are both greater than 8.

The divergence of $\int_0^\infty dx/h(x)$ is equivalent to the divergence of $\sum_{k=1}^\infty 1/h(k)$, and the growth condition on $h$ and (5.7) imply that this is its turn is equivalent to the divergence of

$$\sum_{k=1}^\infty \frac{1}{h(x_k)} \qquad \text{or} \qquad \sum_{k=1}^\infty a_k .$$

We now proceed as we did in Theorem 5.2, and choose a nonincreasing sequence $(b_k)_1^\infty$ such that $\sum_{k=1}^\infty a_k b_k^{1-\alpha}$ diverges and $\sum_{k=1}^\infty a_k b_k$ converges, and put

$$f(x) = \begin{cases} a_k c^{k(1-\alpha)} & \text{on} \quad [c^{-k}, c^{-k}(1+b_k)], \\ 0 & \text{elsewhere.} \end{cases}$$

Then we obtain, just as in the proof of Theorem 5.2, that $\sum' |c_n(f)| \, |n|^{\alpha-1}$ diverges. Using the growth properties of $h$ we see also

$$\int_0^1 |f(x)|^{1/(1-\alpha)} (h(\log^+ |f|))^{\alpha/(1-\alpha)} \, dx \leqslant \sum_{k=1}^\infty a_k^{1/(1-\alpha)} b_k (h(x_k))^{\alpha/(1-\alpha)} = \sum_{k=1}^\infty a_k b_k .$$

Since this series is convergent, our theorem is proved.

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